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SOME ELEMENTARY THEOREMS CONCERNING THE STEADY FLOW OF ELECTRICITY IN SOLID CONDUCTORS

BY B. O. PEIRCE

ALTHOUGH many familiar mathematical truths were first discovered by the study of physical phenomena, and many propositions, the analytical proofs of which are somewhat long and involved, are nearly self-evident when regarded from the point of view of their physical consequences, it is, nevertheless, true that the analytical treatment of a theorem almost always throws new light upon it and shows clearly what its limitations are, if not in what respects it may be stated more generally. It is sometimes true also that reasoning which seems satisfactory enough to a person who has had long experience in making physical measurements, is not so convincing to a student of Mathematics whose knowledge of Physics is largely theoretical. For these reasons it has seemed to me worth while to discuss briefly, from their mathematical side, a few simple theorems concerning the steady flow of electricity in massive conductors; some of these, as being self-evident, are stated without proof in most treatises on the subject; others are rarely, if ever, mentioned.

It is usual to assume* that in an isotropic conductor, K, which is carrying a steady current of electricity, the current vector q has everywhere the direction of the electrostatic field, F, due to all the free electricity in existence, and that the strength of the current at any point is equal to λ times the strength of F at the point, where λ is an intrinsically positive scalar function of the space coordinates called the *specific conductivity* of K. The value of λ is determined by the physical nature of K and by the temperature; the specific conductivity of a homogeneous conductor has the same value at all points of the conductor. The fact that after a steady current has been established in a conductor kept at a uniform temperature, there is no growing accumulation of free electricity in any portion of the conductor leads to the equation of continuity

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* Riemann-Weber: *Die partiellen Differentialgleichungen der mathematischen Physik*, §§ 162, 163.

Webster: *The theory of electricity and magnetism*, chap. VIII, part II.

Maxwell: *A treatise on electricity and magnetism*, vol. 1, part II, chap. VIII.

Gray: *Absolute measurements in electricity and magnetism*, vol. 1, pp. 161-164.

Mascart et Joubert: *Leçons sur l'électricité et le magnétisme*, §§ 198-242.

Curry: *Theory of electricity and magnetism*, § XXII.

$$\text{Divergence } q = 0,$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where u, v, w are the components of the current, parallel to the coordinate axes.

If V is the electrostatic potential function and if X, Y, Z are the components of the electrostatic field,

$$u = \lambda \cdot X = -\lambda \cdot \frac{\partial V}{\partial x}, \quad v = \lambda \cdot Y = -\lambda \cdot \frac{\partial V}{\partial y}, \quad w = \lambda \cdot Z = -\lambda \cdot \frac{\partial V}{\partial z};$$

and the equation of continuity expresses the fact that the divergence of λ times the gradient vector of V is zero; that is

$$\frac{\partial}{\partial x}\left(\lambda \cdot \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \cdot \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \cdot \frac{\partial V}{\partial z}\right) = 0. \tag{2}$$

We shall find it convenient to represent the first member of (2) by the symbol $\Omega(\lambda, V)$.

At the common surface S_{12} of two conductors K_1, K_2 which are carrying a steady current of electricity, the continuity of flow leads to the equation

$$\lambda_1 \cdot \frac{\partial V}{\partial n_1} + \lambda_2 \cdot \frac{\partial V}{\partial n_2} = 0, \tag{3}$$

where λ_1, λ_2 are the specific conductivities of the conductors, and n_1, n_2 represent the directions of the normals to S_{12} at any point on it drawn into K_1 and K_2 respectively. S_{12} may or may not be an equipotential surface; in any case, according to Volta's theory, if the two conductors are made of different materials, there is, in general, a discontinuity of electrostatic potential at their common surface of contact. The magnitude (E_{12}) of this discontinuity, that is, the sudden increase in the value of the potential function at a moving point P as P leaves K_1 and enters K_2, is the same for every point of S_{12} and depends upon the materials of which the conductors are made and upon their temperatures.

The surface of separation (S) of a dielectric and a conductor which is carrying a steady current of electricity has in general an electrostatic charge: at any point of such a surface the derivative of the potential function taken in the direction of the normal pointing into the conductor is zero, but the derivative of the potential function outside the conductor, taken in the direction of the normal pointing into the dielectric, is not usually zero. The equipotential surfaces within the conductor meet S at right angles; the equipotential surfaces without the conductor meet S in general obliquely.

If a conductor which contains within itself no source or sink of electricity is carrying a steady current, current filaments enter the conductor through some portion or portions of its surface called the *anode* or the *anodes*, and current filaments leave the conductor through some other portion or portions of the surface called the *kathode* or the *kathodes*. The derivative of the potential function in the direction of the interior normal to the surface of the conductor is negative at every point of an anode, positive at every point of a kathode.

If every point of a portion (A) of the surface of a simple conductor be kept at the same potential (V_A), and every point of another portion (B) at another potential (V_B) while the rest of the surface abuts on an insulating medium, A and B are *equipotential electrodes*; and if under these circumstances a steady current of strength C passes from A to B the resistance r of the conductor is defined to be $(V_A - V_B)/C$.

An amount of heat equal to $(V_A - V_B)C$ or C^2r is set free in the conductor every second by the current. If n indicates a normal to the surface of the conductor pointing inwards, the strength of the current, C, is equal to the surface integral of $-\lambda \cdot \partial V/\partial n$, taken over A, or to the surface integral of $+\lambda \cdot \partial V/\partial n$, taken over B; and the Joulean heat is equal to

$$-\int\int V \cdot \lambda \cdot \frac{\partial V}{\partial n} \cdot dS, \tag{4}$$

taken over A *and* B, or, since this will give the same result, over the entire surface of the conductor. Everywhere within the conductor $\Omega(\lambda, V)$ vanishes and a simple application of Green's Theorem gives for the Joulean heat the volume integral, taken through the conductor, of λ times the square of the strength of the electrostatic field, or

$$\int\int\int \lambda \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right\} d\tau; \tag{5}$$

this integral extended over the space occupied by a conductor K may be denoted by the symbol

$$J(\lambda, V)_K.$$

If the first of an open chain of conductors has an equipotential electrode

A kept at potential V_A, and the last an equipotential electrode B kept at the lower potential V_B, if, moreover, the algebraic sum of the discontinuities of the potential function at the common surfaces of consecutive conductors (counting a sudden *increase* on the way from A to B as positive) is E, the Joulean heat set free per second within the conductors which form the chain is

$$(V_A - V_B + E)C$$

and this is equal to $J(\lambda, V)$ extended throughout the space occupied by all the conductors. This expression takes no account of the thermal phenomena at the junctions.

If at every point of a portion of the boundary of a region R in space, the value of a function V is given, and at every point of the rest of the boundary the value of the interior-normal derivative of V, and if $\Omega(\lambda, V)$ vanishes within R where λ is a given positive point function which, with its space derivatives of the first order, is finite within R, then V, if it exists, is unique. A problem concerning the steady flow of electricity in a conductor is considered solved, therefore, when a function V has been somehow discovered which is found on trial to satisfy the prescribed conditions for the potential function at the surface of the conductor, and to obey within the conductor the requirement $\Omega(\lambda, V) = 0$.

If over a portion of the boundary S of a given space region R, in which the functions V and V' with their derivatives of the first and second orders are finite, these functions have equal values at every point, while over the remainder of S their normal derivatives vanish; and if within S, $\Omega(\lambda, V) = 0$ and $\Omega'(\lambda, V') \neq 0$, where λ is a positive point function finite with its first derivatives in R: then $J(\lambda, V')_R$ is greater than $J(\lambda, V)_R$.

If there is no flux across a given surface within or at the boundary of a conductor which is carrying a steady current, the surface is called a *current surface* or a *surface of flow*.

A conductor is said to be *given* when its dimensions and its specific conductivity are given; the surface conditions of a conductor which is carrying a steady current are said to be the same at two different times when the electrodes and the values of the potential function at every point of the electrodes are the same. Unless the contrary is stated, the electrodes of a conductor are usually assumed to be equipotential.

The surfaces of equal conductivity $(\lambda = k)$ in a conductor may coincide with the equipotential surfaces; if they happen to do so in any given case, a change in the dimensions or position of an electrode will generally destroy the coincidence.

The flow within an open chain formed of given conductors with given discontinuities of potential at the given surfaces of contact of the successive conductors in the chain, is evidently determined when the value of either the potential function or its normal derivative is given at every point of the free surface of the chain.

If the problem of electrical flow in a given conductor under given surface conditions has been solved, it is possible to write down immediately the value which the potential function in the conductor would have had under the same superficial conditions if the specific conductivity had been changed in certain ways; that is, the solution of one problem in Electrokinematics gives the solutions of certain other related problems. For example: the potential function in a conductor under given surface conditions has the same value as it would have under the same superficial conditions if the specific conductivity, $\lambda = \phi(x,y,z)$, were everywhere increased to m times its old value; the heat developed in the conductor per second would be increased m-fold by the change of conductivity and the resistance of the whole conductor would be decreased in the same ratio.

Given the potential function (V) in any homogeneous conductor which is carrying a steady current under fixed surface conditions, the specific conductivity can be changed in many different ways without changing the value of the potential function; indeed λ may have any value which will make $\Omega(\lambda, V)$ vanish within the conductor. Since V is harmonic, the equation $\Omega(\lambda, V) = 0$ can be written in the form

$$\frac{\partial V}{\partial x} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \frac{\partial V}{\partial z} \cdot \frac{\partial \lambda}{\partial z} = 0. \tag{6}$$

So that λ, if not constant, may be equal to any point function which yields level surfaces perpendicular to the equipotential surfaces.

If the potential function V' in a non-homogeneous conductor is to have the same level surfaces as the potential function V of a homogeneous conductor of the same dimensions under the same surface conditions, we may write $V' = f(V)$ and determine λ from the equation $\Omega(\lambda, V') = 0$, or

$$\lambda \cdot f'(V) \cdot \nabla^2(V) + f'(V) \cdot \left(\frac{\partial V}{dx} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \frac{\partial V}{\partial z} \cdot \frac{\partial \lambda}{\partial z}\right) + \lambda \cdot h_V^2 \cdot f''(V) = 0, \tag{7}$$

where h_V is the gradient of V or the strength of the field within the homogeneous conductor. Now the normal derivative* of the logarithm of λ, taken with respect to V, *i. e.* $[D_V(\log\lambda)]$, is equal to

$$\left(\frac{\partial V}{\partial x}\cdot\frac{\partial\lambda}{\partial x}+\frac{\partial V}{\partial y}\cdot\frac{\partial\lambda}{\partial y}+\frac{\partial V}{\partial z}\cdot\frac{\partial\lambda}{\partial z}\right)\Big/\left(\lambda\cdot h_V^2\right), \tag{8}$$

and V is harmonic, so that (7) reduces to the form

$$\frac{f''(V)}{f'(V)} = -[D_V(\log\lambda)]. \tag{9}$$

If the function V be used to define one of a set of orthogonal curvilinear coordinates (U, V, W), we learn from integrating equation (9) that $\lambda\,.\,f'(V)$ must be independent of V and is, therefore, either an absolute constant or a function of U and W. The surface conditions require that V and $f(V)$ shall be equal at each electrode: otherwise f is wholly arbitrary. If, therefore, (to consider a simple application of what precedes) we have the solution of a problem of steady flow within a homogeneous conductor, then the potential function, under the same superficial conditions, in a stratified conductor which has the same dimensions as the other but a specific conductivity represented by any desired function (ϕ) of the old potential function (V), is of the form $M\cdot F(V) + N$, where $F(V)$ is any integral of the reciprocal of $\phi(V)$ and M, N are constants so chosen as to satisfy the surface conditions. Although the reciprocal of $\phi(V)$ must be integrable, ϕ itself need not be everywhere continuous.

If a conductor which carries a steady current is not homogeneous the potential function (V) within it is not generally harmonic, indeed it is usually true that no harmonic function exists which has the same level surfaces as V. If in such a case the dimensions of the conductor are given as well as the potential function V within it, so that the superficial conditions are determined, the specific conductivity is evidently not determined. The function λ need only satisfy the equation $\Omega(\lambda, V) = 0$, or

$$\lambda\cdot\nabla^2(V) + \lambda\cdot h_V^2\,[D_V(\log\lambda)] = 0, \tag{10}$$

or

$$[D_V\,(\log\lambda)] = -\frac{\nabla^2 V}{h_V^2} \tag{11}$$

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* Peirce: *The Newtonian potential function*, p. 116; *A short table of integrals*, p. 106.

Unless Lamé's condition* for isothermal parameters happens to be satisfied, the second member of (11) usually involves U, V, and W.

If V is not harmonic, but if the λ and V surfaces coincide, λ is a function of V and the condition $\Omega(\lambda, V) = 0$ shows that the ratio of $\nabla^2(V)$ to h^2_V is expressible as a function of V; it is possible, therefore, to find a harmonic function which is expressible in terms of V alone.

We may write equation (2) in the form $\dfrac{\nabla^2(V)}{h^2_V} = -\dfrac{h_\lambda \cdot \cos(\lambda, V)}{\lambda \cdot h_V}$ and assert that if the λ and V surfaces are neither coincident nor orthogonal, V cannot be harmonic.

If the steady flow of electricity in a conductor is defined by a potential function the level surfaces of which are neither coincident with nor orthogonal to the surfaces of constant λ, it is not possible to change the specific conductivity to a new value expressible in terms of the old one — except by altering its value at every point in a constant ratio — if under the same surface conditions, the potential function is to be unchanged. This follows immediately from a comparison of the equations $\Omega(L, V) = 0$ and $\Omega(\lambda, V) = 0$, in which L is supposed to be a function (ϕ) of λ; for if we multiply the second of these by $\phi'(\lambda)$ and subtract it from the first, it appears that both can be satisfied only if

$$\nabla^2(V)[\phi(\lambda) - \lambda \,.\, \phi'(\lambda)] = 0, \tag{12}$$

and, since V is not harmonic, the quantity in square brackets must vanish. If, however, the new value of the specific conductivity be not necessarily a function of the old value alone, we have according to (11),

$$[D_V \log L] = [D_V \log\lambda\,] \text{ or } L = \lambda \cdot F(U, W).$$

If in a conductor which is carrying a steady current, the equipotential surfaces and the surfaces of constant conductivity coincide, so that $\lambda = \phi(V)$ where V is the potential function, then

$$\Omega[\phi(V), V] = 0: \tag{13}$$

if with the same surface conditions the conductivity be changed to λ', a function of the old λ alone, $[\chi(\lambda) = \chi\phi(V) = \psi(V)]$, then the new potential function, V', will be a function of the old one $[V' = f(V)]$, and we shall have

$$\Omega[\psi(V), f(V)] = 0. \tag{14}$$

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* Lamé: *Leçons sur les coordonnées curvilignes*, p. 31; *Leçons sur les fonctions inverses*, p. 5. Somoff-Ziwet: *Theoretische Mechanik*, vol. 1, p. 113, and p. 128.

Equations (13) and (14) when expanded yield the relation

$$\frac{f''(V)}{f'(V)} + \frac{\psi'(V)}{\psi(V)} = \frac{\phi'(V)}{\phi(V)} = -\frac{\nabla^2(V)}{h^2_V}; \tag{15}$$

whence

$$\phi = e^{-\int\frac{\nabla^2(V)dV}{h^2}}, \tag{16}$$

$$\log[f'(V)\cdot\psi(V)] = \log[K\cdot\phi(V)], \tag{17}$$

$$V' = f(V) = K\int\frac{\phi(V)}{\psi(V)}dV + L. \tag{18}$$

In equation (18) which gives the new potential function in terms of the old one, K and L are constants which, after a value has been assigned to the indefinite integral, are to be chosen so as to satisfy the surface conditions.

If V is the potential function in a conductor which has two equipotential electrodes A and B, when these are kept at potentials V_A and V_B, respectively, the value of the potential function when the electrodes are kept at potentials V'_A, V'_B is the linear function of V given by the equation

$$V' = \frac{V'_A - V'_B}{V_A - V_B}\cdot V + \frac{V'_B V_A - V'_A V_B}{V_A - V_B}, \tag{19}$$

for $\Omega\,(\lambda, V') = 0$, and V' satisfies the surface conditions.

Let $V = V_1$, $V = V_2$ be the equations of two equipotential surfaces S_1, S_2 which cut completely across a conductor which is carrying a steady current of electricity in such a manner that the equipotential surfaces are surfaces of constant conductivity so that $\lambda = \phi(V)$. If then the conductivity of the portion of the conductor between S_1 and S_2 be changed so as to become $\lambda' = \psi(V)$, while the surface conditions are unchanged, we may consider the new conductor as a chain of three separate conductors, each of which has two equipotential electrodes. We must assume in general that there is a discontinuity of conductivity at S_1 and at S_2, and that accompanying this (if not in consequence of it) there are discontinuities $(E_1, -E_2)$ of the potential function at these surfaces. The surfaces A and S_1 which in the original conductor were equipotential are still equipotential in the new one though the value of the potential function on S_1 has been changed; it is evident therefore from the statement in the last paragraph that the new potential function in this portion of the conductor must be a linear function, $MV + N$, of the old potential function. Similar reasoning shows that the new potential function in the portion of the conductor which lies between S_2 and B must be of the form $QV + T$, and the potential function in the conductor between S_1 and S_2 of the form $K . f(V) + L$, where $f(V)$ satisfies equation (18).

The six constants M, N, K, L, Q, T are to be determined from the equations

$$
\begin{aligned}
&MV_1 + N + E_1 = K \cdot f(V_1) + L, && K_1 \cdot f(V_2) + L = QV_2 + T + E_2,\\
&MV_A + N = V_A, && QV_B + T = V_B,\\
&M\lambda_1 = K \cdot \lambda'_1 [f'(V)]_{V_1}, && Q \cdot \lambda_2 = K \cdot \lambda'_2 [f'(V)]_{V_2}.
\end{aligned}
$$

If in accordance with equation (17) we put $f'(V) = \lambda/\lambda'$, it appears that

$$M = K = Q = \frac{V_A - V_B + E_1 - E_2}{\Delta},$$

$$N = \frac{V_A\,[f(V_1) - f(V_2) + V_2 - V_1 + E_2 - E_1]}{\Delta},$$

$$T = \frac{V_B\,[f(V_1) - f(V_2) + V_2 - V_1 + E_2 - E_1]}{\Delta},$$

$$L = \frac{(V_B + E_2)\,[f(V_1) + V_A - V_1] - (V_A + E_1)\,[f(V_2) + V_B - V_2]}{\Delta},$$

where $$\Delta = f(V_1) - f(V_2) + V_2 - V_1 + V_A - V_B. \qquad (20)$$

If V is the potential function in a homogeneous conductor of conductivity λ, which has two equipotential electrodes A, B, and if $V = V_0$ is the equation of an equipotential surface S_0 which separates the conductor into two parts, then if the portion of the conductor between S_0 and B changes its conductivity to the uniform value λ'_1 the new potential function, V', will have the value $MV + N$ between A and S_0 and the value $QV + T$ between S_0 and B. We may assume that there will be a discontinuity of potential, E, at S_0. Since $MV_A + N = V_A$, $QV_B + T = V_B$, $QV_0 + T = MV_0 + N + E$, $\lambda M = \lambda' Q$, we have

$$M = \frac{\lambda'}{\lambda} \cdot Q = \frac{\lambda'(V_A - V_B + E)}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A},$$

$$N = \frac{V_A\,[V_B\,(\lambda' - \lambda) + V_0\,(\lambda - \lambda') - \lambda' E]}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A},$$

$$T = \frac{V_B\left[V_0(\lambda - \lambda') + V_A(\lambda' - \lambda) - \lambda E\right]}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A} \tag{21}$$

If one of two conductors (K_1, K_2) which abut at a common surface S_0, has an equipotential electrode A kept at potential V_A and the other an equipotential electrode B kept at potential V_B, and if V is the potential function within the compound conductor when the sudden increase in V in crossing S_0 from K_1 to K_2 is E; then the potential function, V', has in the two conductors the values

$$\frac{(E' + V_A - V_B)\,V + V_A(E - E')}{E + V_A - V_B},$$

$$\frac{(E' + V_A - V_B)\,V + V_B(E - E')}{E + V_A - V_B}, \tag{22}$$

when E' is the discontinuity of potential at S_0 and the other conditions are the same.

If a conductor R has three equipotential electrodes A, B, C kept at three different potentials, we cannot speak of the resistance* of the conductor, though the flow of electricity within it is determined when its conductivity is known, and the development of heat in it per second is perfectly definite. If we have found the values V', V'' of the potential function in R for two distinct sets of surface conditions:

(1) when A, B, C are kept at potentials V'_A, V'_B, V'_C respectively, and
(2) when they are kept at potentials V''_A, V''_B, V''_C;

the potential function when the electrodes are kept at potentials V_A, V_B, V_C is usually of the form

$$V = LV' + MV'' + N; \tag{23}$$

since $\Omega(\lambda, V) = 0$ within R, and at every point of its surface outside of the electrodes $\partial V/\partial n = 0$; if, therefore, L, M, N can be made to satisfy the equations

$$\begin{aligned} LV'_A + MV''_A + N &= V_A, \\ LV'_B + MV''_B + N &= V_B, \\ LV'_C + MV''_C + N &= V_C, \end{aligned} \tag{24}$$

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* Maxwell: *A treatise on electricity and magnetism*, §306.

the statement just made is justified. The procedure fails, however, if

$$\begin{vmatrix} V'_A & V''_A & 1 \\ V'_B & V''_B & 1 \\ V'_C & V''_C & 1 \end{vmatrix} = 0, \tag{25}$$

for then

$$V'' = \frac{(V''_A - V''_B)\,V' + V'_A V''_B - V''_A V'_B}{V'_A - V'_B}, \tag{26}$$

the equipotential surfaces are the same in both solutions, and the cases are not really distinct.

Since in general $\Omega(\lambda, V') = 0$, and $\Omega(\lambda, V'') = 0$ within R, we may learn from Green's Theorem that

$$\iint \lambda \cdot V' \cdot \frac{\partial V''}{\partial n} \cdot dS = \iint \lambda \cdot V'' \cdot \frac{\partial V'}{\partial n} \cdot dS, \tag{27}$$

where the integrations are to be extended over the surface of R. If then I'_A, I'_B, I'_C represent the fluxes inward across the electrodes in the first case and I''_A, I''_B, I''_C the corresponding quantities in the second case,

$$V'_A \cdot I''_A + V'_B \cdot I''_B + V'_C \cdot I''_C = V''_A \cdot I'_A + V''_B \cdot I'_B + V''_C \cdot I'_C, \tag{28}$$

where

$$I'_A + I'_B + I'_C = I''_A + I''_B + I''_C = 0. \tag{29}$$

If A, B, C are given equipotential portions of the free surface of an open chain of conductors of given conductivities with given discontinuities of potential at the surfaces of contact of successive members of the chain; and if V', V'', V are the potential functions within the chain, when A, B, C are kept at potentials V'_A, V'_B, V'_C; V''_A, V''_B, V''_C; V_A, V_B, V_C, respectively; then V is a linear function of V' and V'' in every conductor, if the solutions are distinct. This treatment of the flow within conductors which have two and three electrodes can be easily extended to the flow in a chain of m given conductors with p given equipotential electrodes.

A change in the specific conductivity of a portion of a conductor which has given electrodes, is usually accompanied by a corresponding change in the resistance of the conductor as a whole and by a change in the distribution of the current in all parts of the conductor, though two or three exceptional cases have been considered in the preceding pages. If V is the potential function in a conductor which has two fixed equipotential electrodes A, B, kept at potentials V_A, V_B, and if the specific conductivity of a portion P of the conductor be changed in any manner, provided that its value is increased at every point of P, the amount of heat developed per second in the whole conductor will be increased and its resistance* decreased. The change in the conductivity of P will either leave the potential function V unchanged or it will change V to V'. In the first case, h_V, the strength of the field, will be unchanged, but the positive quantity λh_V^2 (the integral of which taken through the conductor measures the rate at which heat is developed in it) is larger in P than it was before and in the remainder of the conductor has its old value, so that the integral will have a greater value than before. If the new potential function, as will generally be the case, differs from the old one, denote their difference by u, so that $V' = V + u$. At every point of either electrode $V = V'$, so that $u = 0$; at every other point of the boundary of the conductor $\partial V/\partial n = 0$, $\partial V'/\partial n = 0$, so that $\partial u/\partial n = 0$; within the conductor $\Omega(\lambda, V) = 0$, $\Omega(\lambda, V') = 0$, so that $\Omega(\lambda, u) = 0$: we have, therefore, [see equation (5)]

$$J(\lambda, V') = J(\lambda, V) + J(\lambda, u) + 2\iiint \lambda\left(\frac{\partial V}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial V}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial V}{\partial z}\cdot\frac{\partial u}{\partial z}\right) d\tau \qquad (30)$$

$$= J(\lambda, V) + J(\lambda, u) - 2\iint \lambda\cdot u\cdot\frac{\partial V}{\partial n}\cdot dS - 2\iiint u\cdot\Omega(\lambda, V)\,d\tau \qquad (31)$$

The integrand of the surface integral in (31) vanishes at every point of the boundary of the conductor, for at the electrodes $u = 0$, and at the surface of separation of the conductor and the surrounding dielectric the normal derivative of V taken inwards must vanish; the integrand of the volume integral vanishes also at every point of the conductor, so that

$$J(\lambda, V') = J(\lambda, V) + J(\lambda, u). \qquad (32)$$

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* Rayleigh: On the theory of resonance, *Phil. Trans.* vol. 161, 1871; *Scientific papers*, vol. 1, p. 51.

Webster: *The theory of electricity and magnetism*, § 175.

Maxwell: *A treatise on electricity and magnetism*, §§ 306, 307.

Now $J(\lambda, u)$ is not zero, for u is not a constant; it is not negative, for its integrand cannot be negative; therefore $J(\lambda, u)$ is positive and

$$J(\lambda, V') > J(\lambda, V);$$

but

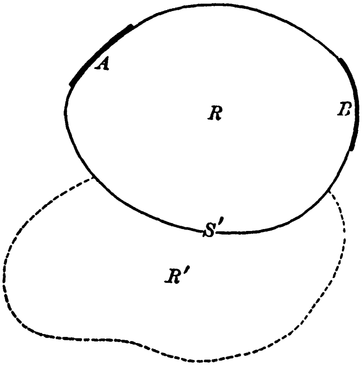
$$J(\lambda', V') > J(\lambda, V');$$

therefore, *a fortiori*,

$$J(\lambda', V') > J(\lambda, V). \qquad (33)$$

The heat developed in the conductor, per second, by a steady current must be equal to the product of the constant $(V_A - V_B)$ and the strength of the current; in the present case, therefore, the current must be greater when the conductivity of P is increased and the resistance of the whole conductor must be less.

If a conductor, R, which has two equipotential electrodes, A and B, and is carrying a steady current be increased in bulk by bringing up to its side another conductor R' which has no electrodes of its own but is in electrical contact with R over a portion S' of its surface outside of A and B, then the composite conductor will always have a smaller resistance than R has alone. Since parallel tangential derivatives of the potential function at points close together but on opposite sides of S' must have equal limits, either S' must have been made equipotential by the coming of R' or there must be a potential gradient in R' near S' and hence a flow of electricity in R' which has no electrodes and must receive any current which it has through current filaments which cross S'. In either case S' cannot continue to be a surface of flow and the po-



tential function in R cannot be the same as it was before R' was brought up. Let V and V' be the old and the new potential functions and write $u = V' - V$; then

$$J(\lambda, V')_R = J(\lambda, V)_R + J(\lambda, u)_R - 2\int\int \lambda \cdot u \cdot \frac{\partial V}{\partial n} \cdot dS$$
$$- 2\int\int\int u \cdot \Omega(\lambda, V) d\tau, \qquad (34)$$

where the volume integral is to be taken through R and the double integral over its surface. Since the integrands vanish, we have

$$J(\lambda, V')_R > J(\lambda, V)_R, \qquad (35)$$

so that the development of heat *in R alone* is greater after the bringing up of R' than before. Heat is also developed in R' and the current in the composite conductor is greater than the original current in R.

If in a conductor R which is carrying a steady current of electricity under given superficial conditions a surface S' not previously equipotential be suddenly forced to become so, by making the specific conductivity at all points on and very near to S' infinite, while the rest of R is unchanged, the resistance of the conductor will be decreased. In this theorem S' may represent a closed surface which shuts in a finite part of the conductor and excludes a part, it may be an open surface which cuts completely across the conductor, or it may be practically a portion of an open surface bounded by a curve wholly within or partly on the surface of the conductor. In any case if S' was not previously equipotential the new potential function, V', must differ from the old one, and if we enclose S' between two parallel surfaces close to it, write $V' - V = u$, and compute the values of $J(\lambda, V')$ and $J(\lambda, V)$ for the space within R modified in this way, we shall get an equation which is formally the same as (35).

If an "infinitely thin, insulating membrane" were suddenly inserted into R so as to coincide with an open or closed surface S'', wholly within R, across which there had previously been a flux, the resistance of the conductor would be increased. If V is the actual potential function and V'' the potential function in the supposed case, we may write $V - V'' = u$, shut in S'' between two parallel surfaces S''_1, S''_2 drawn very close to it, find the values of $J(\lambda, V)$, $J(\lambda, V')$

for the region within the conductor thus modified, and suppose S_1'' and S_2'' to approach each other indefinitely. We learn that

$$J(\lambda, V) = J(\lambda, V'') + J(\lambda, u) - 2\int\!\!\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS$$
$$- 2\int\!\!\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS_1'' - 2\int\!\!\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS_2'', \qquad (36)$$

where the first surface integral is to be extended over the outer boundary of the conductor. Since the derivative of V'' in a direction normal to S'' is equal to zero at every point on either side of this surface, the integrals vanish and $J(\lambda, V) = J(\lambda, V'') + J(\lambda, u)$, so that

$$J(\lambda, V'') < J(\lambda, V). \qquad (37)$$

If an interior portion, R_0, of a conductor R which is carrying a steady current of electricity under given surface conditions could be displaced by matter having at every point the same specific conductivity as before but of such a nature that at every point of S_0, the boundary of R_0, the potential function just within this surface should be greater than the potential function just without by a given constant amount E, then the resistance of the conductor as a whole would not be changed. Suppose the old and the new potential functions to be V and V_0, shut S_0 in between two near parallel surfaces S_0', S_0'', find the value of $J(\lambda, V_0 - V)$ extended through all the conductor which is not between S_0', S_0'', and determine the limit of this value as these surfaces approach each other indefinitely. If $u = V_0 - V$, u is zero at every point of either electrode and $\partial u/\partial n$ is zero at every point of the boundary of the conductor outside of the electrodes. The value of u at any point just inside S_0 differs from the value of u at a neighboring point just outside S_0 by E. It is evident from equation (3) that the normal derivatives of V and V_0 are continuous at S_0.

$$J(\lambda, u) = -\int\!\!\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS - \int\!\!\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS_0' - \int\!\!\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS_0'', \quad (38)$$

where the first integral is to be extended over the outer boundary of the conductor, and n indicates the directions of a normal pointing into the field of the volume integral represented by the first member. The integrand of the first integral vanishes; the limit of the sum of the other two integrals is

$$-E\int\int\lambda\cdot\frac{\partial u}{\partial n}\cdot dS_0, \tag{39}$$

where n points into the region bounded by S_0. By a simple application of Green's Theorem, we have, therefore,

$$J(\lambda, u)_R = E\int\int\int\Omega(\lambda, u)\,d\tau, \tag{40}$$

where the volume integral is to be extended through the space bounded by S_0; but in this region $\Omega(\lambda,u) = 0$, so that the integral $J(\lambda,u)_R$ is equal to zero, and u must be constant. It is evident that since $u = 0$ at an electrode, it has the same value everywhere inside the conductor outside S_0; within S_0, it has everywhere the value E. At every point within R, the field is the same in direction and strength after the change and before it, so that

$$J(\lambda, V) = J(\lambda, V_0). \tag{41}$$

Given a conductor of any shape which has two equipotential electrodes, A and B, kept at potentials V_A, V_B and which is carrying a steady current of electricity, it is easy to see that if one electrode (say A) be enlarged so as to occupy an additional portion A' of the surface of the conductor, the resistance of the conductor will be decreased. If V is the old potential function and V' the new one, $\partial V/\partial n$ is zero on A', V' is equal to V_A on A'. If $u = V' - V$,

$$J(\lambda, V') = J(\lambda, V) + J(\lambda,u) - 2\int\int\lambda\,.\,u\frac{\partial V}{\partial n}dS - 2\int\int\int u\cdot\Omega(\lambda, V)\,d\tau \tag{42}$$

$$= J(\lambda, V) + J(\lambda,u). \tag{43}$$

$J(\lambda,u)$ is positive since u is not constant, therefore

$J(\lambda, V')$ is greater than $J(\lambda, V)$.

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